Example: Using the shortcut rules, calculate

$$[xe^x \sin(x)]'.$$

$$[xe^x \sin(x)]' = (xe^x)'\sin(x) + xe^x\cdot(\sin(x))'$$

$$= [(x)'e^x + x(e^x)']\sin(x) + xe^x(\sin(x))'$$

$$= (x)'e^x\sin(x) + x(e^x)'\sin(x) + xe^x(\sin(x))'$$

$$= 1\cdot e^x\sin(x) + xe^x\sin(x) + xe^x\cos(x)$$

$$= \boxed{e^x[\sin(x) + x\sin(x) + x\cos(x)]}.$$

Generalized Product Rule:

$$[f(x)g(x)h(x)\cdots z(x)]'$$

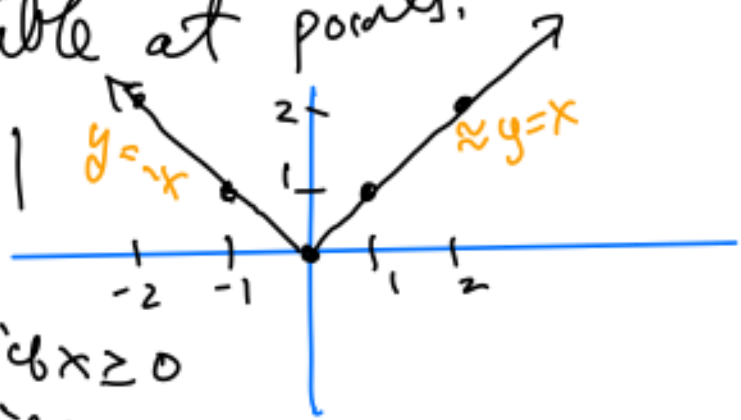
$$= f'(x)g(x)h(x)\cdots z(x) + f(x)g'(x)h(x)\cdots z(x) +$$

$$\cdots + f(x)g(x)h(x)\cdots z'(x)$$

Examples of functions that are not differentiable at points.

Ⓐ $y = |x|$

$A(x)$



$$y = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

$A(x)$ is not differentiable at $x=0$

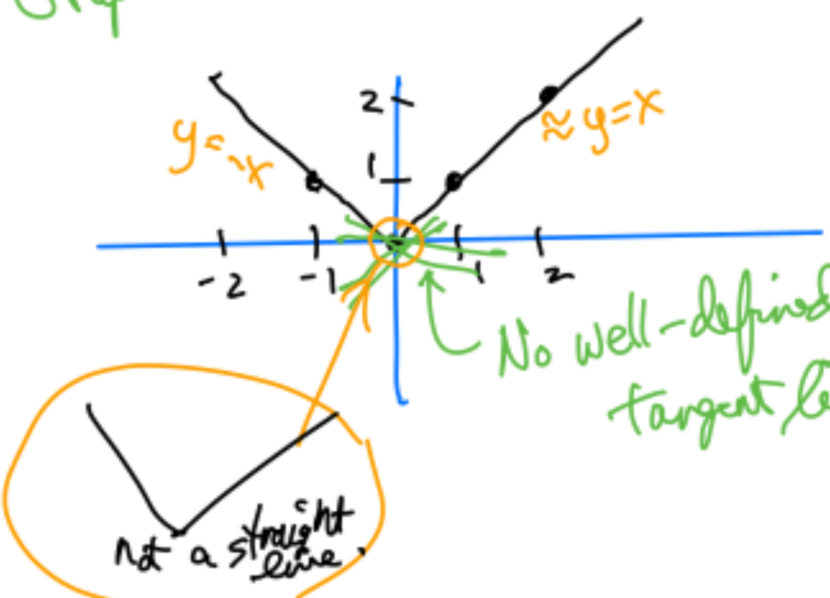
$A'(x) = 1$ if $x > 0$ $\quad$ $A'(0)$ does not exist.

$A'(x) = -1$ if $x < 0$

Why? $A'(0) = \lim\limits_{h\to 0} \dfrac{A(0+h) - A(0)}{h} = \lim\limits_{h\to 0} \dfrac{|h| - |0|}{h}$

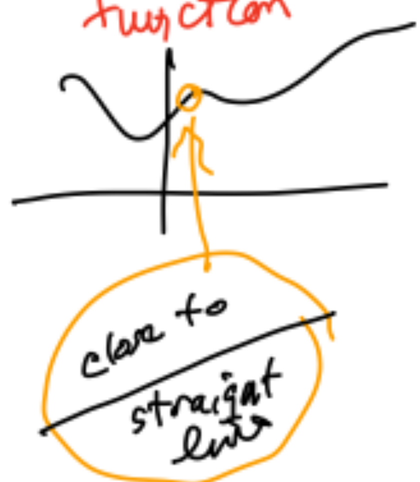
$= \lim\limits_{h\to 0} \dfrac{|h|}{h} \leftarrow \left.\begin{array}{l} \lim\limits_{h\to 0^+} \dfrac{|h|}{h} = 1 \\ \lim\limits_{h\to 0^-} \dfrac{|h|}{h} = -1 \end{array}\right\} \lim\limits_{h\to 0} \dfrac{|h|}{h}$ DNE.

Graphical reason.



No well-defined tangent line.

differentiable function

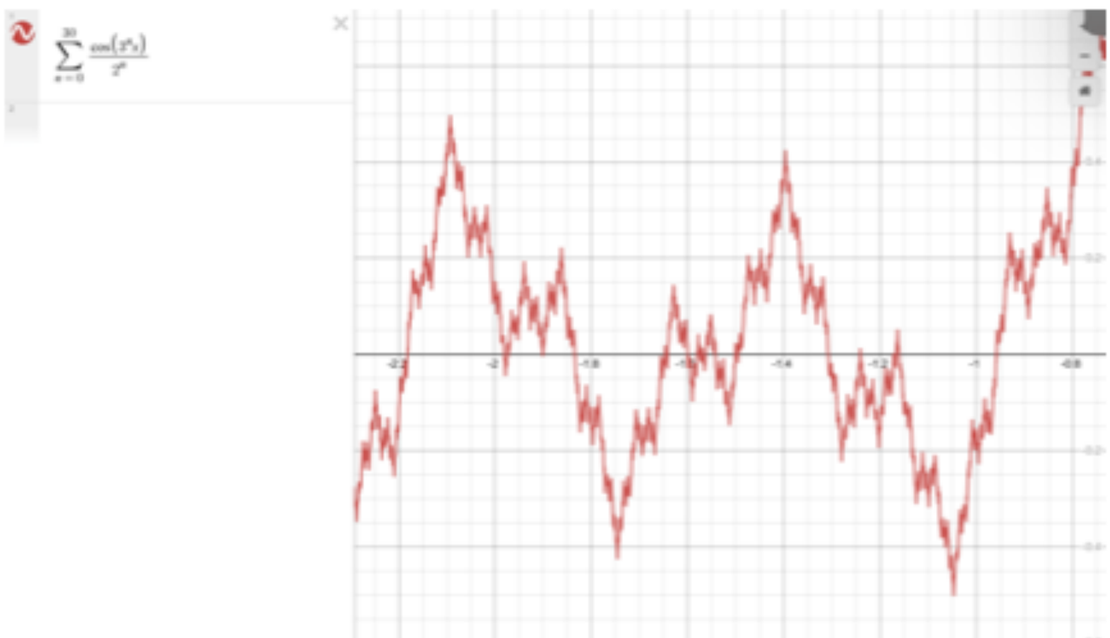


Ⓑ Crazy example

Weierstrass function.

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(3^n x)}{2^n}$$

$$= \underset{n=0}{\frac{\cos(x)}{1}} + \underset{n=1}{\frac{\cos(3x)}{2}} + \underset{n=2}{\frac{\cos(9x)}{4}} + \frac{\cos(27x)}{8} + \cdots$$



This function f(x) is continuous at every x but is not differentiable at every single x.

(i.e. $f'(x)$ DNE at each x).

New shortcut rule:

The Quotient Rule.

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$ ⟸ if $g(x) \neq 0$, f, g differentiable at x.

$$\left[\frac{Hi}{Ho}\right]' = \frac{Ho\, dHi - Hi\, dHo}{Ho\, Ho}$$

Pf. Observe that

$$f(x) = g(x) \cdot \frac{f(x)}{g(x)}.$$

$$\Rightarrow f'(x) = g'(x) \cdot \frac{f(x)}{g(x)} + g(x) \cdot \left[\frac{f(x)}{g(x)}\right]'$$

by the Product Rule

$$\Rightarrow g(x) f'(x) = g'(x) f(x) + (g(x))^2 \cdot \left(\frac{f(x)}{g(x)}\right)'$$

$$g(x) f'(x) - g'(x) f(x) = (g(x))^2 \left(\frac{f}{g}\right)'$$

$$\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = \left(\frac{f(x)}{g(x)}\right)' \quad \blacksquare$$